Linear Algebra and its Applications 436 (2012) 4251–4260

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Derivable maps and derivational points ☆

Zhidong Pan

Department of Mathematics, Saginaw Valley State University, University Center, MI 48710, USA

ARTICLE INFO

Article history:
Received 9 September 2011
Accepted 22 January 2012
Available online 25 February 2012

Submitted by P. Šemrl

AMS classification:
Primary: 47B47
47L35

Keywords:
Derivable maps
Derivation
Nest

ABSTRACT

For an algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$, let $L(\mathcal{A}, \mathcal{M})$ be the set of all linear maps from $\mathcal{A}$ to $\mathcal{M}$. A map $\delta \in L(\mathcal{A}, \mathcal{M})$ is called *derivable at* $C \in \mathcal{A}$ if $\delta(A)B + A\delta(B) = \delta(C)$, for all $A, B \in \mathcal{A}$ with $AB = C$. We call an element $C \in \mathcal{A}$ a *derivational point* of $L(\mathcal{A}, \mathcal{M})$ if $\forall\, \delta \in L(\mathcal{A}, \mathcal{M})$ the condition δ is derivable at C implies δ is a derivation. We characterize derivable maps by means of Peirce decompositions and determine derivational points for some general bimodules. As a special case, we see that for a nest algebra $\mathcal{A}$ on a Hilbert space H, every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A}, B(H))$.

1. Introduction

Let $\mathcal{A}$ be a unital algebra, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, $L(\mathcal{A}, \mathcal{M})$ be the set of all linear maps from $\mathcal{A}$ to $\mathcal{M}$. A map $\delta \in L(\mathcal{A}, \mathcal{M})$ is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$, for all $A, B \in \mathcal{A}$. A map $\delta \in L(\mathcal{A}, \mathcal{M})$ is called *derivable at* $C \in \mathcal{A}$ if $\delta(A)B + A\delta(B) = \delta(C)$, for all $A, B \in \mathcal{A}$ with $AB = C$. We call an element $C \in \mathcal{A}$ a *derivational point* of $L(\mathcal{A}, \mathcal{M})$ if $\forall\, \delta \in L(\mathcal{A}, \mathcal{M})$ the condition δ is derivable at C implies δ is a derivation. A map $\delta \in L(\mathcal{A}, \mathcal{M})$ is called a Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$, for all $A \in \mathcal{A}$.

Many authors have studied derivable maps. The problem to characterize maps that are derivable at 0 is part of a general problem to characterize maps (more generally, additive maps from rings to bimodules) by their actions on zero-products, see, e.g. [1–5,7–9,11]. Jordan derivations have been extensively studied and it is easy to check that all Jordan derivations are derivable at the unit I of the algebra; thus they are part of maps that are derivable at I. Recently, some have become interested in maps that are derivable at elements other than 0 or I, see, e.g. [10,12–16]. In this paper, we give

☆ This work was completed with the support the Ruth and Ted Braun Fellowship from the Saginaw Community Foundation.
E-mail address: pan@svsu.edu

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.01.027

some general characterizations of derivable maps. The characterizations will be used to determine derivational points for some general bimodules. Immediate consequences of our results generalize several results in the literature.

Let X be a complex Banach space and $B(X)$ be the set of all bounded linear operators on X. A *subspace lattice* $\mathcal{L}$ of X is a collection of closed subspaces of X containing 0 and X such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ of X, let $alg\mathcal{L} = \{A \in B(X) : AL \subseteq L,\ \forall L \in \mathcal{L}\}$. A totally ordered subspace lattice is called a *nest* . If $\mathcal{L}$ is a nest, then $alg\mathcal{L}$ is called a *nest algebra*., see [6] for more on nest algebras. When X is a Hilbert space, we change it to H. For Hilbert spaces, we disregard the distinction between a closed subspace and the orthogonal projection onto it.

In Section 2, characterizations of derivable maps are given in terms of Peirce decompositions. In Section 3, we determine derivational points of some general bimodules using the characterizations obtained in Section 2. In Section 4, as an immediate consequence of our results, we see that if $\mathcal{A}$ is a nest algebra on a Hilbert space H then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A},\ B(H))$.

2. Peirce decomposition and derivable maps

For a unital algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$, we denote $\mathcal{A}' = \{M \in \mathcal{M} :\ AM = MA,\ \forall A \in \mathcal{A}\}$. A map $\delta \in L(\mathcal{A}, \mathcal{M})$ is called a *generalized derivation* if there exists an $M_\delta \in \mathcal{A}'$ such that $\delta(AB) = \delta(A)B + A\delta(B) - M_\delta AB$, for all $A, B \in \mathcal{A}$. For any $M \in \mathcal{M}$, define $\delta_M(A) = MA - AM,\ \forall A \in \mathcal{A}$. Clearly δ_M is a derivation from $\mathcal{A}$ into $\mathcal{M}$. Such derivations are called *inner derivations*.

For any idempotent $E_1 \in \mathcal{A}$, let $E_2 = I - E_1$. For $i, j = 1, 2$, define $\mathcal{A}_{ij} = E_i\mathcal{A}E_j$; this gives the Peirce decomposition of $\mathcal{A}$: $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. For any $A \in \mathcal{A}$, let $A_{ij} = E_iAE_j$. Similarly, we define $\mathcal{M}_{ij} = E_i\mathcal{M}E_j$. We say $\mathcal{A}_{ij}$ is *left faithful* with respect to $\mathcal{M}$ if for any $M \in \mathcal{M}$, the condition $M\mathcal{A}_{ij} = \{0\}$ implies $ME_i = 0$ and we say $\mathcal{A}_{ij}$ is *right faithful* with respect to $\mathcal{M}$ if the condition $\mathcal{A}_{ij}M = \{0\}$ implies $E_jM = 0$. We say $\mathcal{A}_{ij}$ is *faithful* with respect to $\mathcal{M}$ if it is both left faithful and right faithful. In this paper, we will always let $P = E_1$ and $Q = E_2 = I - P$ for simpler notations.

Throughout Sections 2 and 3, we will assume $\mathcal{A}$ is a unital algebra over a field $\mathcal{F}$ with $|\mathcal{F}| \geqslant 4$, $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule, and $\mathcal{A}$ has a nontrivial idempotent $P = E_1 \in \mathcal{A}$ such that the corresponding Peirce decomposition has the following property: Every element of $\mathcal{A}_{11}$ is a linear combination of invertible elements of $\mathcal{A}_{11}$ and every element of $\mathcal{A}_{22}$ is a linear combination of invertible elements of $\mathcal{A}_{22}$. Algebras that satisfy these assumptions include all finite-dimensional unital algebras over an algebraically closed field and all unital Banach algebras.

Proposition 2.0. *Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$ with $|\mathcal{F}| \geqslant 3$. For any fixed $u, v, w \in \mathcal{V}$, define $p(t) = ut^2 + vt + w,\ \forall t \in \mathcal{F}$. If $p(t) = 0$ has three distinct solutions in $\mathcal{F}$ then $u = v = w = 0$.*

Theorem 2.1. *For any $C \in \mathcal{A}$ such that $C_{21} = 0$, if $\Delta \in L(\mathcal{A}, \mathcal{M})$ is derivable at C then there exists a $\delta \in L(\mathcal{A}, \mathcal{M})$ such that $\Delta - \delta$ is an inner derivation and the following hold*

(*a*) $P\delta(P)A_{12} = A_{12}\delta(Q),\ \forall A_{12} \in \mathcal{A}_{12}$.
(*b*) $\delta(A_{12})A_{12} = A_{12}\delta(A_{12}) = 0,\ \forall A_{12} \in \mathcal{A}_{12}$.
If $\mathcal{A}_{12}$ is left faithful with respect to $\mathcal{M}$ then
(*c*) $P\delta(P)A_{11} = A_{11}P\delta(P),\ \forall A_{11} \in \mathcal{A}_{11}$, *i.e.* $P\delta(P) \in \mathcal{A}'_{11}$.
(*d*) $\delta|_{\mathcal{A}_{11}}$ *is a generalized derivation from $\mathcal{A}_{11}$ to $\mathcal{M}$.*
(*e*) $\delta|_{\mathcal{A}_{11}}$ *is derivable at C_{11}.*
If $\mathcal{A}_{12}$ is right faithful with respect to $\mathcal{M}$ then
(*f*) $\delta(Q)A_{22} = A_{22}\delta(Q),\ \forall A_{22} \in \mathcal{A}_{22}$, *i.e.* $\delta(Q) \in \mathcal{A}'_{22}$.
(*g*) $\delta|_{\mathcal{A}_{22}}$ *is a generalized derivation from $\mathcal{A}_{22}$ to $\mathcal{M}_{22}$.*
(*h*) $\delta|_{\mathcal{A}_{22}}$ *is derivable at C_{22}.*

Proof. Let $M = P\Delta(Q)Q - Q\Delta(Q)P$ and define $\delta(A) = \Delta(A) - (MA - AM),\ \forall A \in \mathcal{A}$. Then δ is derivable at any $G \in \mathcal{A}$ iff Δ is derivable at G; moreover $\delta(Q) \in \mathcal{M}_{11} + \mathcal{M}_{22}$. Write $C = C_{11} + C_{12} + C_{22}$. Fix any $A_{11} \in \mathcal{A}_{11}$ that is invertible in $\mathcal{A}_{11}$ with $A_{11}^{-1} \in \mathcal{A}_{11}$ and $Z_{22},\ W_{22} \in \mathcal{A}_{22}$ such that $Z_{22}W_{22} = C_{22}$.

Note that we can take any W_{22} that is invertible in $\mathcal{A}_{22}$ with $W_{22}^{-1} \in \mathcal{A}_{22}$ and $Z_{22} = C_{22}W_{22}^{-1}$ to satisfy $Z_{22}W_{22} = C_{22}$.

For any $0 \neq t \in \mathcal{F}$ and $A_{12} \in \mathcal{A}_{12}$, a routine computation shows

$$[A_{11} + t(A_{11}A_{12} + Z_{22})][(A_{11}^{-1}C - A_{12}W_{22}) + t^{-1}W_{22}] = C.$$

Since δ is derivable at C,

$$\begin{aligned}\delta(C) &= \delta[A_{11} + t(A_{11}A_{12} + Z_{22})]\,[(A_{11}^{-1}C - A_{12}W_{22}) + t^{-1}W_{22}] \\ &\quad + [A_{11} + t(A_{11}A_{12} + Z_{22})]\,\delta[(A_{11}^{-1}C - A_{12}W_{22}) + t^{-1}W_{22}] \\ &= t\,\{\delta(A_{11}A_{12} + Z_{22})(A_{11}^{-1}C - A_{12}W_{22}) + (A_{11}A_{12} + Z_{22})\delta(A_{11}^{-1}C - A_{12}W_{22})\} \\ &\quad + t^{-1}\,[\delta(A_{11})W_{22} + A_{11}\delta(W_{22})]\; + \delta(A_{11})(A_{11}^{-1}C - A_{12}W_{22}) \\ &\quad + A_{11}\delta(A_{11}^{-1}C - A_{12}W_{22}) + \delta(A_{11}A_{12} + Z_{22})W_{22} + (A_{11}A_{12} + Z_{22})\delta(W_{22}).\end{aligned}$$

Since t is arbitrary, by Proposition 2.0 we get

$$\{\delta(A_{11}A_{12} + Z_{22})(A_{11}^{-1}C - A_{12}W_{22}) + (A_{11}A_{12} + Z_{22})\delta(A_{11}^{-1}C - A_{12}W_{22})\} = 0 \tag{2.1}$$

$$[\delta(A_{11})W_{22} + A_{11}\delta(W_{22})] = 0 \tag{2.2}$$

and

$$\begin{aligned}\delta(C) &= \delta(A_{11})(A_{11}^{-1}C - A_{12}W_{22}) + A_{11}\delta(A_{11}^{-1}C - A_{12}W_{22}) \\ &\quad + \delta(A_{11}A_{12} + Z_{22})W_{22} + (A_{11}A_{12} + Z_{22})\delta(W_{22}) \\ &= \delta(A_{11})A_{11}^{-1}C + A_{11}\delta(A_{11}^{-1}C) + \delta(Z_{22})W_{22} + Z_{22}\delta(W_{22}) \\ &\quad - \delta(A_{11})A_{12}W_{22} - A_{11}\delta(A_{12}W_{22}) + \delta(A_{11}A_{12})W_{22} + A_{11}A_{12}\delta(W_{22}).\end{aligned}$$

Since A_{12} in the above equation is arbitrary, we have

$$\delta(C) = \delta(A_{11})A_{11}^{-1}C + A_{11}\delta(A_{11}^{-1}C) + \delta(Z_{22})W_{22} + Z_{22}\delta(W_{22}).$$

and

$$-\delta(A_{11})A_{12}W_{22} - A_{11}\delta(A_{12}W_{22}) + \delta(A_{11}A_{12})W_{22} + A_{11}A_{12}\delta(W_{22}) = 0 \tag{2.3}$$

By Eq. (2.2), $\delta(A_{11})Q + A_{11}\delta(Q) = 0$. Since $\delta(Q) \in \mathcal{M}_{11} + \mathcal{M}_{22}$, we have

$$\delta(A_{11})Q = A_{11}\delta(Q) = 0 \tag{2.4}$$

Thus, $\delta(A_{11})W_{22} = 0$, which gives $A_{11}\delta(W_{22}) = 0$ when applied to Eq. (2.2). In particular,

$$P\delta(W_{22}) = 0 \tag{2.5}$$

Taking $A_{11} = P$ in Eq. (2.3) we get

$$P\delta(A_{12}W_{22}) = \delta(A_{12})W_{22} + A_{12}\delta(W_{22}) - \delta(P)A_{12}W_{22} \tag{2.6}$$

Multiplying P from the right of Eq. (2.6) gives

$$P\delta(A_{12}W_{22})P = A_{12}\delta(W_{22})P \tag{2.7}$$

By Eq. (2.5) and $\delta(Q) \in \mathcal{M}_{11} + \mathcal{M}_{22}$, $\delta(Q) \in \mathcal{M}_{22}$. Taking $W_{22} = Q$ in (2.7), we see

$$P\delta(A_{12})P = A_{12}\delta(Q)P = 0 \tag{2.8}$$

Multiplying P from the left of Eq. (2.6), setting $W_{22} = Q$, and applying Eq. (2.8), we get

$$P\delta(A_{12}) = P\delta(A_{12})Q + A_{12}\delta(Q) - P\delta(P)A_{12} = P\delta(A_{12}) + A_{12}\delta(Q) - P\delta(P)A_{12}.$$

It follows that

$$P\delta(P)A_{12} = A_{12}\delta(Q) \tag{2.9}$$

This proves (a).
Rewriting Eq. (2.1), we get

$$\begin{aligned}0 = {} & \delta(Z_{22})A_{11}^{-1}C + Z_{22}\delta(A_{11}^{-1}C) - [\delta(A_{11}A_{12})A_{12}W_{22} + A_{11}A_{12}\delta(A_{12}W_{22})] \\ & + \{\delta(A_{11}A_{12})A_{11}^{-1}C + A_{11}A_{12}\delta(A_{11}^{-1}C) - \delta(Z_{22})A_{12}W_{22} - Z_{22}\delta(A_{12}W_{22})\}.\end{aligned}$$

Replacing A_{12} with kA_{12}, for any $k \in \mathcal{F}$, we see

$$\begin{aligned}0 = {} & \delta(Z_{22})A_{11}^{-1}C + Z_{22}\delta(A_{11}^{-1}C) - k^2[\delta(A_{11}A_{12})A_{12}W_{22} + A_{11}A_{12}\delta(A_{12}W_{22})] \\ & + k\{\delta(A_{11}A_{12})A_{11}^{-1}C + A_{11}A_{12}\delta(A_{11}^{-1}C) - \delta(Z_{22})A_{12}W_{22} - Z_{22}\delta(A_{12}W_{22})\}.\end{aligned}$$

Since k is arbitrary, by Proposition 2.0 we have

$$[\delta(A_{11}A_{12})A_{12}W_{22} + A_{11}A_{12}\delta(A_{12}W_{22})] = 0.$$

Multiplying P from the left of this equation and applying Eq. (2.8), we get

$$A_{11}A_{12}\delta(A_{12}W_{22}) = 0 \tag{2.10}$$

It follows

$$\delta(A_{11}A_{12})A_{12}W_{22} = 0 \tag{2.11}$$

Taking $A_{11} = P$ and $W_{22} = Q$ in (2.10) and (2.11) completes the proof of (b).
Taking $W_{22} = Q$ in Eq. (2.3), we get

$$\delta(A_{11}A_{12})Q = \delta(A_{11})A_{12} + A_{11}\delta(A_{12}) - A_{11}A_{12}\delta(Q) \tag{2.12}$$

Multiplying Q from the left of Eq. (2.12) and setting $A_{11} = P$, we have

$$Q\delta(A_{12})Q = Q\delta(P)A_{12} \tag{2.13}$$

For any $U_{11} \in \mathcal{A}_{11}$ that is invertible in $\mathcal{A}_{11}$, by Eq. (2.8),

$$A_{11}\delta(U_{11}A_{12})Q = A_{11}\delta(U_{11}A_{12}).$$

Using this with Eq. (2.12), we get

$$\begin{aligned}\delta(A_{11}U_{11}A_{12})Q &= A_{11}\delta(U_{11}A_{12}) + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q) \\ &= A_{11}\delta(U_{11}A_{12})Q + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q) \\ &= A_{11}[\delta(U_{11})A_{12} + U_{11}\delta(A_{12}) - U_{11}A_{12}\delta(Q)] \\ &\quad + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q).\end{aligned}$$

Thus

$$\delta(A_{11}U_{11}A_{12})Q = A_{11}\delta(U_{11})A_{12} + A_{11}U_{11}\delta(A_{12}) + \delta(A_{11})U_{11}A_{12} - 2A_{11}U_{11}A_{12}\delta(Q).$$

On the other hand, by Eq. (2.12) again,

$$\delta(A_{11}U_{11}A_{12})Q = \delta(A_{11}U_{11})A_{12} + A_{11}U_{11}\delta(A_{12}) - A_{11}U_{11}A_{12}\delta(Q).$$

From the last two equations, we get

$$\delta(A_{11}U_{11})A_{12} = A_{11}\delta(U_{11})A_{12} + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q) \tag{2.14}$$

By Eqs. (2.9) and (2.14),

$$\delta(A_{11}U_{11})A_{12} = A_{11}\delta(U_{11})A_{12} + \delta(A_{11})U_{11}A_{12} - P\delta(P)A_{11}U_{11}A_{12}.$$

If $\mathcal{A}_{12}$ is left faithful then from the last equation and Eq. (2.4), we get

$$\delta(A_{11}U_{11}) = A_{11}\delta(U_{11}) + \delta(A_{11})U_{11} - P\delta(P)A_{11}U_{11} \tag{2.15}$$

The hypothesis that every element of $\mathcal{A}_{11}$ is a linear combination of invertible elements of $\mathcal{A}_{11}$ implies that Eq. (2.15) holds for all $A_{11}, U_{11} \in \mathcal{A}_{11}$.

Taking $U_{11} = P$ in Eq. (2.15), and applying (2.4) gives

$$\delta(A_{11}) = A_{11}\delta(P) + \delta(A_{11})P - P\delta(P)A_{11} = A_{11}\delta(P) + \delta(A_{11}) - P\delta(P)A_{11}.$$

Thus $A_{11}P\delta(P) = P\delta(P)A_{11}$, which proves (c).

Item (d) follows from (c) and Eq. (2.15) directly.

Multiplying Q from the left of Eq. (2.15) and setting $A_{11} = P$, we have the following which will be used in Section 3

$$Q\delta(U_{11}) = Q\delta(P)U_{11} \tag{2.16}$$

Since $IC = C$ and δ is derivable at C, it follows $\delta(I)C = 0$. Combining this with Eq. (2.5), we have $P\delta(P)C_{11} = P\delta(P)CP = P\delta(I)CP = 0$. For any $A_{11}, U_{11} \in \mathcal{A}_{11}$ such that $A_{11}U_{11} = C_{11}$, by Eq. (2.15) we get $\delta(A_{11}U_{11}) = A_{11}\delta(U_{11}) + \delta(A_{11})U_{11}$. This proves (e).

For any $A_{22} \in \mathcal{A}_{22}$, by Eq. (2.9), $A_{12}\delta(Q)A_{22} = P\delta(P)A_{12}A_{22} = A_{12}A_{22}\delta(Q)$. If $\mathcal{A}_{12}$ is right faithful, $A_{12}\delta(Q)A_{22} = A_{12}A_{22}\delta(Q)$ implies $Q\delta(Q)A_{22} = QA_{22}\delta(Q)$. Thus $\delta(Q)A_{22} = A_{22}\delta(Q)$. This proves (f).

For any invertible $V_{22} \in \mathcal{A}_{22}$, replacing A_{12} with $A_{12}V_{22}$ in Eq. (2.6) gives

$$P\delta(A_{12}V_{22}W_{22}) = \delta(A_{12}V_{22})W_{22} + A_{12}V_{22}\delta(W_{22}) - \delta(P)A_{12}V_{22}W_{22}.$$

Multiplying P from the left of the last equation and applying Eq. (2.6) again, we have

$$\begin{aligned} P\delta(A_{12}V_{22}W_{22}) &= P\delta(A_{12}V_{22})W_{22} + A_{12}V_{22}\delta(W_{22}) - P\delta(P)A_{12}V_{22}W_{22} \\ &= [\delta(A_{12})V_{22} + A_{12}\delta(V_{22}) - \delta(P)A_{12}V_{22}]W_{22} \\ &\quad + A_{12}V_{22}\delta(W_{22}) - P\delta(P)A_{12}V_{22}W_{22}. \end{aligned}$$

On the other hand, replacing W_{22} with $V_{22}W_{22}$ in Eq. (2.6) gives

$$P\delta(A_{12}V_{22}W_{22}) = \delta(A_{12})V_{22}W_{22} + A_{12}\delta(V_{22}W_{22}) - \delta(P)A_{12}V_{22}W_{22}.$$

From the last two equations, we see

$$A_{12}\delta(V_{22}W_{22}) = A_{12}\delta(V_{22})W_{22} + A_{12}V_{22}\delta(W_{22}) - P\delta(P)A_{12}V_{22}W_{22}.$$

Applying Eq. (2.9), we get

$$A_{12}\delta(V_{22}W_{22}) = A_{12}\delta(V_{22})W_{22} + A_{12}V_{22}\delta(W_{22}) - A_{12}\delta(Q)V_{22}W_{22}.$$

If $\mathcal{A}_{12}$ is right faithful, the last equation and (2.5) give us

$$\delta(V_{22}W_{22}) = \delta(V_{22})W_{22} + V_{22}\delta(W_{22}) - \delta(Q)V_{22}W_{22} \tag{2.17}$$

The hypothesis that every element of $\mathcal{A}_{22}$ is a linear combination of invertible elements of $\mathcal{A}_{22}$ implies that Eq. (2.17) holds for all $V_{22}, W_{22} \in \mathcal{A}_{22}$.

By Eqs. (2.7) and (2.8), $A_{12}\delta(W_{22})P = 0$. If $\mathcal{A}_{12}$ is right faithful, we have $Q\delta(W_{22})P = 0$; from this and Eq. (2.5), we get

$$\delta(W_{22}) \in \mathcal{M}_{22} \tag{2.18}$$

Now (g) follows from (f), and Eqs. (2.17) and (2.18).

Since $CI = C$ and δ is derivable at C, it follows $C\delta(I) = 0$; combining this with (f) and (2.4), we have $\delta(Q)C_{22} = C_{22}\delta(Q) = QC_{22}\delta(Q)Q = QC\delta(I)Q = 0$. For any $V_{22}, W_{22} \in \mathcal{A}_{22}$ such that $V_{22}W_{22} = C_{22}$, by Eq. (2.17) we get $\delta(V_{22}W_{22}) = \delta(V_{22})W_{22} + V_{22}\delta(W_{22})$. This proves (h). $\square$

3. Peirce decomposition and derivational points

Theorem 3.1. *Let $C \in \mathcal{A}$ be such that $QC = 0$, and $\forall\, M \in \mathcal{M}$, $MC = 0$ implies $MP = 0$. If $\mathcal{A}_{12}$ is faithful with respect to $\mathcal{M}$, then C is a derivational point of $L(\mathcal{A}, \mathcal{M})$.*

Proof. Suppose $\delta \in L(\mathcal{A}, \mathcal{M})$ is derivable at C. Subtracting an inner derivation from δ, if necessary, we can assume δ satisfies all the properties discussed in Theorem 2.1.

First we show $\forall\, A_{ij} \in \mathcal{A}_{ij},\ \delta(A_{ij}) \in M_{ij},\ \forall\, i, j = 1, 2$.

By (2.18), $\delta(A_{22}) \in \mathcal{M}_{22}$.

Since $IC = C$ and δ is derivable at C, we get $\delta(I)C = 0$. By the hypothesis, $\delta(I)C = 0$ implies $\delta(I)P = 0$; thus, by (2.4), $\delta(P) = \delta(P)P = \delta(I)P = 0$. By (2.16), $Q\delta(A_{11}) = Q\delta(P)A_{11} = 0,\ \forall\, A_{11} \in \mathcal{A}_{11}$. Combining with (2.4), we have

$$\delta(A_{11}) \in \mathcal{M}_{11} \tag{3.1}$$

By (2.9), $A_{12}\delta(Q) = 0,\ \forall\, A_{12} \in \mathcal{A}_{12}$. Since $\mathcal{A}_{12}$ is faithful, $Q\delta(Q) = \delta(Q) = 0$.

By (2.8), $P\delta(A_{12})P = 0$. By (2.13), $Q\delta(A_{12})Q = Q\delta(P)A_{12} = 0$.

Since $QC = 0$, $(I + A_{12})C = C$. Since δ is derivable at C and $\delta(I) = \delta(P) + \delta(Q) = 0$, it follows $A_{12}\delta(C) + \delta(A_{12})C = 0$. Multiplying Q from the left of the equation gives $Q\delta(A_{12})C = 0$. Thus $Q\delta(A_{12})P = 0$, by the hypothesis. Therefore

$$\delta(A_{12}) \in \mathcal{M}_{12} \tag{3.2}$$

Since $C = C_{11} + C_{12}$, $P\delta(C) = \delta(C)$ by (3.1) and (3.2). Since $P(C + A_{21}) = C$ and δ is derivable at C, it follows

$$P\delta(A_{21}) = 0 \tag{3.3}$$

Since $(P + A_{12})(C - A_{12}A_{21} - A_{12} + A_{21} + Q) = C$ and δ is derivable at C,

$$\delta(P + A_{12})(C - A_{12}A_{21} - A_{12} + A_{21} + Q) + (P + A_{12})\delta(C - A_{12}A_{21} - A_{12} + A_{21} + Q) = \delta(C).$$

Applying (3.1) $--$(3.3) to the above equation, we get

$$\delta(A_{12})A_{21} - \delta(A_{12}A_{21}) + A_{12}\delta(A_{21}) = 0 \tag{3.4}$$

Multiplying Q from the right of (3.4) gives $A_{12}\delta(A_{21})Q = 0$. Since $\mathcal{A}_{12}$ is faithful, $Q\delta(A_{21})Q = 0$; combining with (3.3) we get $\delta(A_{21}) \in \mathcal{M}_{21}$.

To show δ is a derivation, it suffices to prove that for any $A_{ij} \in \mathcal{A}_{ij},\ A_{kl} \in \mathcal{A}_{kl}$,

$$\delta(A_{ij}A_{kl}) = \delta(A_{ij})A_{kl} + A_{ij}\delta(A_{kl}) \tag{3.5}$$

We will label each case as Case (ij, kl). Since $\delta(A_{ij}) \in M_{ij},\ \forall\, i, j = 1, 2$, we only need to prove Eq. (3.5) for $j = k$.

Case $(11, 11)$ follows from Eq. (2.15).

Case $(11, 12)$ follows from Eq. (2.12).

Case $(12, 21)$ follows from Eq. (3.4).

Case $(12, 22)$ follows from Eq (2.6).

Case $(22, 22)$ follows from Eq (2.17).

It remains to show Cases $(21, 11)$, $(21, 12)$, and $(22, 21)$.

Applying Case $(12, 21)$, we get

$$\delta(A_{12}A_{21}A_{11}) = \delta(A_{12})A_{21}A_{11} + A_{12}\delta(A_{21}A_{11}) \tag{3.6}$$

Using Cases (11, 11) and (12, 21), we have

$$\begin{aligned}\delta(A_{12}A_{21}A_{11}) &= \delta(A_{12}A_{21})A_{11} + A_{12}A_{21}\delta(A_{11}) \\ &= \delta(A_{12})A_{21}A_{11} + A_{12}\delta(A_{21})A_{11} + A_{12}A_{21}\delta(A_{11})\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), $A_{12}\delta(A_{21}A_{11}) = A_{12}\delta(A_{21})A_{11} + A_{12}A_{21}\delta(A_{11})$. Since $\mathcal{A}_{12}$ is faithful, we get $\delta(A_{21}A_{11}) = \delta(A_{21})A_{11} + A_{21}\delta(A_{11})$, completing the proof of Case (21, 11).

Applying Case (12, 22), we get

$$\delta(A_{12}A_{21}A_{12}) = \delta(A_{12})A_{21}A_{12} + A_{12}\delta(A_{21}A_{12}) \tag{3.8}$$

Using Cases (11, 12) and (12, 21), we have

$$\begin{aligned}\delta(A_{12}A_{21}A_{12}) &= \delta(A_{12}A_{21})A_{12} + A_{12}A_{21}\delta(A_{12}) \\ &= \delta(A_{12})A_{21}A_{12} + A_{12}\delta(A_{21})A_{12} + A_{12}A_{21}\delta(A_{12})\end{aligned} \tag{3.9}$$

By (3.8) and (3.9), $A_{12}\delta(A_{21}A_{12}) = A_{12}\delta(A_{21})A_{12} + A_{12}A_{21}\delta(A_{12})$. Since $\mathcal{A}_{12}$ is faithful, we get $\delta(A_{21}A_{12}) = \delta(A_{21})A_{12} + A_{21}\delta(A_{12})$, completing the proof of Case (21, 12).

Applying Case (12, 21), we get

$$\delta(A_{12}A_{22}A_{21}) = \delta(A_{12})A_{22}A_{21} + A_{12}\delta(A_{22}A_{21}) \tag{3.10}$$

Using Cases (12, 21) and (12, 22), we have

$$\begin{aligned}\delta(A_{12}A_{22}A_{21}) &= \delta(A_{12}A_{22})A_{21} + A_{12}A_{22}\delta(A_{21}) \\ &= \delta(A_{12})A_{22}A_{21} + A_{12}\delta(A_{22})A_{21} + A_{12}A_{22}\delta(A_{21})\end{aligned} \tag{3.11}$$

By (3.10) and (3.11), $A_{12}\delta(A_{22}A_{21}) = A_{12}\delta(A_{22})A_{21} + A_{12}A_{22}\delta(A_{21})$. Since $\mathcal{A}_{12}$ is faithful, we get $\delta(A_{22}A_{21}) = \delta(A_{22})A_{21} + A_{22}\delta(A_{21})$, completing the proof of Case (22, 21). $\square$

Corollary 3.2. *Suppose $\mathcal{A}_{12}$ is faithful with respect to $\mathcal{M}$. If $C \in \mathcal{A}$ such that $QC = 0$ and C_{11} is invertible in $\mathcal{A}_{11}$ then C is a derivational point of $L(\mathcal{A}, \mathcal{M})$; in particular, P is a derivational point of $L(\mathcal{A}, \mathcal{M})$.*

If $\mathcal{B}$ is an algebra containing $\mathcal{A}$, then $\mathcal{B}$ is an $\mathcal{A}$-bimodule with respect to the multiplication and addition of $\mathcal{B}$. In this case we have

Theorem 3.3. *Suppose $\mathcal{A}_{12}$ is faithful with respect to $\mathcal{B}$ and $\mathcal{A}_{21} = \{0\}$. If $\mathcal{A}' = \mathcal{F}I$ then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A}, \mathcal{B})$.*

Proof. Suppose $0 \neq C \in \mathcal{A}$ and $\delta \in L(\mathcal{A}, \mathcal{B})$ is derivable at C, subtracting an inner derivation from δ if necessary, we can assume δ satisfies the properties of Theorem 2.1.

First we claim that for any $B \in \mathcal{B}$,

$$T_{12}BT_{12} = 0, \ \forall\, T_{12} \in \mathcal{A}_{12} \ \text{ implies } \ QBP = 0 \tag{3.12}$$

Suppose $T_{12}BT_{12} = 0$, $\forall\, T_{12} \in \mathcal{A}_{12}$. For any non-zero $A_{12}, T_{12} \in \mathcal{A}_{12}$, we have $A_{12}BA_{12} = 0$, $T_{12}BT_{12} = 0$, and $(A_{12} + T_{12})B(A_{12} + T_{12}) = 0$. It follows that

$$A_{12}BT_{12} + T_{12}BA_{12} = 0 \tag{3.13}$$

For any $A_{11} \in \mathcal{A}_{11}$, replacing A_{12} in Eq. (3.13) with $A_{11}A_{12}$ gives

$$A_{11}A_{12}BT_{12} + T_{12}BA_{11}A_{12} = 0 \tag{3.14}$$

Multiplying A_{11} from the left of Eq. (3.13) gives

$$A_{11}A_{12}BT_{12} + A_{11}T_{12}BA_{12} = 0 \tag{3.15}$$

By Eqs. (3.14) and (3.15), we see $T_{12}BA_{11}A_{12} = A_{11}T_{12}BA_{12}$. Since A_{12} is arbitrary and $\mathcal{A}_{12}$ is faithful, we have

$$T_{12}BPA_{11} = A_{11}T_{12}BP \tag{3.16}$$

Similarly, for any $A_{22} \in \mathcal{A}_{22}$, replacing A_{12} in Eq. (3.13) with $A_{12}A_{22}$ gives

$$A_{12}A_{22}BT_{12} + T_{12}BA_{12}A_{22} = 0 \tag{3.17}$$

Multiplying A_{22} from the right of Eq. (3.13) gives

$$A_{12}BT_{12}A_{22} + T_{12}BA_{12}A_{22} = 0 \tag{3.18}$$

By Eqs. (3.17) and (3.18), we see $A_{12}A_{22}BT_{12} = A_{12}BT_{12}A_{22}$. Since A_{12} is arbitrary and $\mathcal{A}_{12}$ is faithful, we have

$$A_{22}QBT_{12} = QBT_{12}A_{22} \tag{3.19}$$

Let $B' = T_{12}BP - QBT_{12}$. It follows from Eqs. (3.13), (3.16), and (3.19) that B' commutes with A_{12}, A_{11}, and A_{22}; so $B' \in \mathcal{A}'$. Thus, there exists a $\lambda \in \mathcal{F}$ such that $B' = \lambda I$. It follows $T_{12}BP = \lambda P$. Now $T_{12}BT_{12} = 0$ gives $\lambda T_{12} = 0$, thus $\lambda = 0$ and $T_{12}BP = 0$. Since T_{12} is arbitrary and $\mathcal{A}_{12}$ is faithful, we have $QBP = 0$, completing the proof of (3.12).

Next, we show $\forall\, A_{ij} \in \mathcal{A}_{ij},\ \delta(A_{ij}) \in \mathcal{B}_{ij},\ \forall\, i, j = 1, 2$.

By our assumption, $\mathcal{A}_{21} = \{0\}$; and clearly $\delta(0) \in \mathcal{B}_{21}$.

For any $0 \neq T_{12} \in \mathcal{A}_{12}$, by Theorem 2.1($b$) we have $T_{12}\delta(T_{12}) = 0$. Multiplying Q from the right and applying Eq. (2.13), we get $T_{12}Q\delta(P)T_{12} = 0$. Thus $Q\delta(P)P = 0$, by (3.12). For any $A_{11} \in \mathcal{A}_{11}$, by Eq. (2.16), $Q\delta(A_{11}) = Q\delta(P)PA_{11} = 0$. Combining this with Eq. (2.4), we get $\delta(A_{11}) \in \mathcal{B}_{11}$.

For any $A_{12},\ T_{12} \in \mathcal{A}_{12}$, by Theorem 2.1($b$) we get $\delta(A_{12})A_{12} = 0$, $\delta(T_{12})T_{12} = 0$, and $\delta(T_{12} + A_{12})(T_{12} + A_{12}) = 0$. It follows that $\delta(A_{12})T_{12} + \delta(T_{12})A_{12} = 0$. Multiplying T_{12} from the left and applying $T_{12}\delta(T_{12}) = 0$, we get $T_{12}\delta(A_{12})T_{12} = 0$. Thus $Q\delta(A_{12})P = 0$, by (3.12). By Eq. (2.13) we see $Q\delta(A_{12})Q = Q\delta(P)PA_{12} = 0$. Together with Eq. (2.8), we have $\delta(A_{12}) \in \mathcal{B}_{12}$.

Finally, by Eq. (2.18), $\delta(A_{22}) \in \mathcal{B}_{22},\ \forall\, A_{22} \in \mathcal{A}_{22}$.

Since $\delta(P) \in \mathcal{B}_{11}$ and $\delta(Q) \in \mathcal{B}_{22}$, by Theorem 2.1($a$)($c$)($f$), $\delta(I) = \delta(P) + \delta(Q)$ commutes with A_{12}, A_{11}, and A_{22}; so $\delta(I) \in \mathcal{A}'$. Thus $\delta(I) = kI$ for some $k \in \mathcal{F}$.

Since δ is derivable at C and $IC = C$, we have $\delta(I)C = 0$; or $kC = 0$. Since $C \neq 0$, $k = 0$. Thus $\delta(P) = \delta(Q) = 0$.

To see δ is a derivation, we only need to verify

$$\delta(A_{ij}A_{kl}) = \delta(A_{ij})A_{kl} + A_{ij}\delta(A_{kl}).$$

We will again label each case as Case (ij, kl). Since $\delta(A_{ij}) \in \mathcal{B}_{ij},\ \forall\, i, j = 1, 2$, we only need to consider cases for $j = k$. Since $\mathcal{A}_{21} = \{0\}$, there are only 4 cases.

Case (11, 11) follows from Eq. (2.15).

Case (11, 12) follows from Eq. (2.12).

Case (12, 22) follows from Eq. (2.6).

Case (22, 22) follows from Eq. (2.17). □

4. Applications to algebras of operators on Banach spaces

Let X^* be the topological dual of X. For any $e \in X$ and $f^* \in X^*$, let $e \otimes f^*$ be the rank-one operator defined by $(e \otimes f^*)x = f^*(x)e,\ \forall\, x \in X$. For any $C \in B(X)$, we use $ran(C)$ to denote the range of C. Note that [10, Theorem 3.1] can be stated in the following slightly more general form, the same proof in [10] will remain intact.

Proposition 4.1 [10, Theorem 3.1]**.** *Let $\mathcal{A}$ be a norm-closed unital subalgebra of $B(X)$ such that $\vee\{x : \forall\, 0 \neq x \otimes f^* \in \mathcal{A}\} = X$. If $\delta \in L(\mathcal{A}, B(X))$ is derivable at some $C \in \mathcal{A}$ and $ran(C)$ is dense in X, then δ is a derivation; i.e. C is a derivational point of $L(\mathcal{A}, B(X))$.*

Remark. Many subalgebras $\mathcal{A}$ of $B(X)$ satisfy $\vee\{x : \forall\, 0 \neq x \otimes f^* \in \mathcal{A}\} = X$, including $\mathcal{J}$-subspace lattice algebras and completely distributive subspace lattice algebras, in particular, nest algebras.

The following generalizes [13, Theorem 4.1].

Corollary 4.2. *Let $\mathcal{N}$ be a nest on a Banach space X and $\mathcal{A} = alg\mathcal{N}$ be the corresponding nest algebra. If $C \in \mathcal{A}$ such that either $ran(C)$ is dense in X or C is injective, then C is a derivational point of $L(\mathcal{A},\ B(X))$.*

Proof. Let $x \mapsto \hat{x}$ be the canonical map from X to X^{**}, then $(x \otimes f^*)^* = f^* \otimes \hat{x}$. A standard routine verification shows $\vee\{x : \forall\, 0 \neq x \otimes f^* \in \mathcal{A}\} = X$ and $\vee\{f^* : \forall\, 0 \neq f^* \otimes \hat{x} \in \mathcal{A}^*\} = X^*$. If $ran(C)$ is dense in X, we can apply Proposition 4.1 directly. If C is injective, then C^* has dense range in X^*. For any $\delta \in L(\mathcal{A},\ B(X))$ that is derivable at C, define a linear map $\delta^* : \mathcal{A}^* \mapsto B(X^*)$ by $\delta^*(A^*) = \delta(A)^*,\ \forall A^* \in \mathcal{A}^*$. It follows that δ^* is derivable at C^*. Thus δ^* is a derivation by Proposition 4.1, this implies δ is a derivation. $\square$

Lemma 4.3. *For a Hilbert space H, every $0 \neq C \in B(H)$ is a derivational point of $L(B(H), B(H))$.*

Proof. If $ran(C)$ is dense in H, the conclusion follows from Proposition 4.1 with $\mathcal{A} = B(H)$.

Suppose $0 \neq C \in B(H)$ and $ran(C)$ is not dense in H. We will apply Theorem 3.1 with $\mathcal{A} = \mathcal{M} = B(H)$. Let P be the orthogonal projection from H onto the closure of $ran(C)$ and $Q = I - P$. Then $QC = 0$, and $\forall\, M \in B(H),\ MC = 0$ implies $MP = 0$. Clearly, $\mathcal{A}_{12} = PB(H)Q$ is faithful, so Theorem 3.1 applies. $\square$

As a consequence of Theorem 3.3, we have

Theorem 4.4. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X and $\mathcal{A} = alg\mathcal{L}$. If there exists a nontrivial idempotent $P \in \mathcal{A}$ such that $ran(P) \in \mathcal{L}$ and $PB(X)(I - P) \subseteq \mathcal{A}$ then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A},\ B(X))$.*

Proof. We will apply Theorem 3.3 with $\mathcal{B} = B(X)$. Let $Q = I - P$. The condition $ran(P) \in \mathcal{L}$ implies $\mathcal{A}_{21} = Q\mathcal{A}P = \{0\}$. The condition $PB(X)Q \subseteq \mathcal{A}$ implies $\mathcal{A}_{12} = PB(X)Q$ is faithful. To see $\mathcal{A}' = \mathbb{C}I$, take any $B \in \mathcal{A}'$, from $BP = PB$ we get $PBQ = QBP = 0$. Thus $B = B_{11} + B_{22}$. For any $x \in ran(P)$ and $f^* \in X^*$, $x \otimes f^*Q \in \mathcal{A}_{12}$. It follows from $Bx \otimes f^*Q = x \otimes f^*QB$ that

$$B_{11}x \otimes f^*Q = x \otimes f^*QB_{22} \tag{4.1}$$

By (4.1), $B_{11}x \in \mathbb{C}x$. Since $x \in ran(P)$ is arbitrary, it follows $B_{11} = \lambda P$ for some $\lambda \in \mathbb{C}$. Combining with (4.1) we have, $x \otimes f^*(\lambda Q - B_{22}) = 0$. Thus $B_{22} = \lambda Q$ and $B = \lambda I$. The conclusion now follows from Theorem 3.3. $\square$

The following generalizes [13, Theorem 3.1].

Corollary 4.5. *Let $\mathcal{N}$ be a nest on a Banach space X and $\mathcal{A} = alg\mathcal{N}$. If there exists a nontrivial idempotent $P \in \mathcal{A}$ such that $ran(P) \in \mathcal{N}$ then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A},\ B(X))$.*

Proof. Let $Q = I - P$, check $PB(X)Q \subseteq \mathcal{A}$ and then apply Theorem 4.4. $\square$

Theorem 4.6. *If $\mathcal{A} = alg\mathcal{N}$ is a nest algebra on a Hilbert space H then every $0 \neq C \in \mathcal{A}$ is a derivational point of $L(\mathcal{A},\ B(H))$.*

Proof. If $\mathcal{N} = \{0, H\}$, then $\mathcal{A} = alg\mathcal{N} = B(H)$, the conclusion follows from Lemma 4.3. If there exists a nontrivial $P \in \mathcal{N}$, the conclusion follows from Corollary 4.5. $\square$

Theorem 4.6 generalizes several results in the literature, including [12, Theorem 2.7, 15, Theorem 3.1, 16, Theorem 3.2].

The author reported some results of this paper, including Theorem 4.6, at the 27th South Eastern Analysis Meeting at the University of Florida on March 17—19, 2011.

References

[1] J. Alaminos, M. Bresar, M. Cerne, J. Extremera, A.R. Villena, Zero product preserving maps on $C^1[0, 1]$, J. Math. Anal. Appl. 347 (2008) 472–481.
[2] J. Alaminos, M. Bresar, J. Extremera, A.R. Villena, Maps preserving zero products, Studia Math. 193 (2009) 131–159.
[3] J. Alaminos, J. Extremera, A.R. Villena, M. Bresar, Characterizing homomorphisms and derivations on C^*-algebras, Proc. Royal Soc. Edinburgh Sect. A 137 (2007) 1–7.
[4] M. Bresar, Characterizing homomorphisms, derivations and multipliers in rings with idempotents, Proc. Royal Soc. Edinburgh Sect. A 137 (2007) 9–21.
[5] M. Chebotar, W. Ke, P. Lee, Maps characterized by action on zero products, Pacific J. Math. 216 (2004) 217–228.
[6] K. Davidson, Nest algebras, Pitman Res. Notes Math. 191 (1988).
[7] J. Hou, X. Qi, Additive maps derivable at some points on $\mathcal{J}$-subspace lattice algebras, Linear Algebra Appl. 429 (2008) 1851–1863.
[8] W. Jing, S. Lu, P. Li, Characterisations of derivations on some operator algebras, Bull. Austral. Math. Soc. 66 (2002) 227–232.
[9] J. Li, Z. Pan, Annihilator-preserving maps multipliers and derivations, Linear Algebra Appl. 432 (2010) 5–13.
[10] J. Li, Z. Pan, On derivable mappings, J. Math. Anal. Appl. 374 (2011) 311–322.
[11] J. Li, Z. Pan, H. Xu, Characterizations of isomorphisms and derivations of some algebras, J. Math. Anal. Appl. 332 (2007) 1314–1322.
[12] F. Lu, Characterizations of derivations and Jordan derivations on Banach algebras, Linear Algebra Appl. 430 (2009) 2233–2239.
[13] X. Qi, J. Hou, Characterizations of derivations of Banach space nest algebras: all-derivable points, Linear Algebra Appl. 432 (2010) 3183–3200.
[14] J. Zhou, Linear mappings derivable at some nontrivial elements, Linear Algebra Appl. 435 (2011) 1972–1986.
[15] J. Zhu, C. Xiong, L. Zhang, All-derivable points in matrix algebras, Linear Algebra Appl. 430 (2009) 2070–2079.
[16] J. Zhu, C. Xiong, R. Zhang, All-derivable points in the algebra of upper triangular matrices, Linear Algebra Appl. 429 (2008) 804–818.